# CALCULATION STUDIES OF THE DYNAMIC CHARACTERISTICS OF SPHERICAL AND TOROIDAL SHELLS

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An algorithm for the calculation of natural vibrations and waves, based on the use of special matrices belonging to the type of the Green matrices (according to the procedure [1]), is realized in this work. Results of calculations for a spherical shell and a toroidal shell are presented and discussed. The frequencies and modes of natural frequencies of gradually increasing tones are demonstrated. The data obtained are used for the calculation of propagating waves for various methods of initial perturbation. Specific features of the distribution of dynamic stresses are discovered.

1. Thin-walled constructional elements are extensively used in aviation, rocketry, and other fields of modern technology. The maintenance of reliability of constructions for progressively increasing intensity of working processes gives rise to a need for more accurate account of the dynamic factors. Due to the absence of effective calculation algorithms for cases that are close to real situation, it is often necessary to use crude models. Thus, when calculating the frequencies and modes of natural vibrations of a toroidal shell, the latter is approximately replaced by a cylindrical shell (see [2]). The distortions arising as a result become notable in the case of large curvatures of the equators. To obtain more accurate results we must work out calculation procedures which more fully take into account the geometrical properties of the objects being investigated.

In [1] a possibility of using matrices belonging to the type of the Green matrices, when calculating the natural frequencies and investigating propagating waves in shells, is noted. Let

$$A\left(\varphi, \vartheta; \frac{\partial}{\partial\varphi}, \frac{\partial}{\partial\vartheta}\right) U = -\frac{\rho}{D}\frac{\partial^2 U}{\partial t^2}, \qquad U = U(\varphi, \vartheta; t) = \begin{pmatrix} U^1 \\ U^2 \\ U^3 \end{pmatrix}, \qquad D = \frac{Eh^3}{12(1-\sigma^2)} \tag{1.1}$$

be the abbreviated (matrix) notation of the system of differential equations determining the vibrations of a certain shell. Here A ($\varphi$, $\vartheta$; $\partial/\partial\varphi$, $\partial/\partial\vartheta$) denotes a matrix of linear differential operators, U is the displacement vector of points of the middle surface of the shell, $\rho$ is the specific weight, D is the cylindrical rigidity, E and $\sigma$ are the Young's modulus and Poisson's ratio respectively, and k is the thickness of the shell.

If G ($\varphi$, $\vartheta$; $\alpha$, $\beta$) is a special matrix of the Green matrix type, for which

$$U(\varphi, \vartheta) = \int_S G(\varphi, \vartheta; \alpha, \beta) F(\alpha, \beta)\, d_{\alpha,\beta}\Omega$$

[F ($\alpha$, $\beta$) is the external surface load], then, as is noted in [1], the frequencies and modes of natural vibrations are determined by an intergral equation of the form

$$U(\varphi, \vartheta) = \lambda^4 \iint G(\varphi, \vartheta; \alpha, \beta)\, U(\alpha, \beta)\, d_{\alpha,\beta}\Omega \tag{1.2}$$

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TABLE 1

| $k$ | $m$ | | | | | |
|---|---|---|---|---|---|---|
| | 0 | 4 | 8 | 12 | 16 | 20 |
| 1 | 0.23756 | 2.18147 | 9.28284 | 21.12483 | 37.68109 | 58.93870 |
| 2 | 0.81827 | 2.90829 | 10.1746 | 22.18600 | 38.91601 | 60.31930 |
| 3 | 1.87075 | 4.58635 | 12.4061 | 24.74807 | 41.88852 | 63.68932 |
| 4 | 4.26306 | 8.04533 | 22.4877 | 47.24211 | 81.65108 | 125.1058 |

where $\lambda^4 = \rho\omega^2/D$, and $\omega$ is the frequency of natural vibrations of the shell.

For computations (1.2) is approximated by a system of linear algebraic equations

$$U(\varphi_j) - \lambda^4 \sum_{i=1}^{N} P_i q_m (\varphi_j, \alpha_i) U(\alpha_i) = 0 \tag{1.3}$$

where $\alpha_i$ and $P_i$ are the quadrature nodes and coefficients respectively. The eigenvalues are determined from the condition

$$\det \{J - \lambda^4 Q\} = 0 \tag{1.4}$$

where J is a unit matrix, while Q is the matrix of the system (1.3). It is convenient to make the intervals of sign change of the left side of the condition (1.4) more accurate by a successive sorting. Thus we achieve a high degree of accuracy in the calculation of the roots of Eq. (1.4).

As is mentioned in [3], the calculation of the eigenfunctions can be carried out according to standard programs used for the solution of systems of linear algebraic equations. Here the system (1.3) is slightly modified (for more details, see [3]).

The Green matrices were calculated for toroidal shells with the parametrication

$$x = (R + a\cos\varphi)\cos\vartheta, \quad y = (R + a\cos\varphi)\sin\vartheta, \quad z = a\sin\varphi$$

with the conditions of [1] fulfilled. These matrices can be used to calculate the frequencies and modes of vibrations.

If we put $\sigma = 0.25$, $h = 1$, $a = 100$, $R = 0$ (a spherical shell), then we obtain the eigenvalues $\lambda^4_{km}$ whose values are presented in Table 1. The modes of natural vibrations corresponding to them are shown in Fig. 1. In the upper row of the figure we have placed the images of the deforming equator. The values of the index m (m = 0, m = 4, m = 8 etc.) increasing from left to right determine the number of the harmonic being specified and the direction of the $\vartheta$ coordinate. The eigenvalues for each given m increase from top to bottom ($\lambda_{1m}, \lambda_{2m}, \ldots, \lambda_{km}$). At the same time an increase in the number of nodes on the meridian takes place. The accuracy of the results is checked by comparison of the data obtained with two approximations of different detail.

2. The eigenvalues and eigenfunctions thus obtained can be used to study propagating waves. Let in addition to (1.1) the initial conditions

$$U(\varphi, \vartheta; t)|_{t=0} = \Phi(\varphi, \vartheta), \qquad \left.\frac{\partial U(\varphi, \vartheta; t)}{\partial t}\right|_{t=0} = \Psi(\varphi, \vartheta) \tag{2.1}$$

hold.

The solution of the problem (2.1) for the system (1.1) according to the procedure of separation of variables, by means of the eigenvalues $\lambda_{km}$ and the eigenfunctions $U_{km}$ determined above, is represented in the form

$$U(\varphi, \vartheta; t) = \sum_{km} U_{km}(\varphi)\,\theta_m(\vartheta)\left(\Phi_{km}\cos\omega_{km}t + \frac{\Psi}{\omega_{km}}\sin\omega_{km}t\right)$$

where $\Phi_{km}$ and $\Psi_{km}$ are the coefficients of the expansion

$$\Phi(\varphi, \vartheta) = \sum_{km} \Phi_{km} U_{km}(\varphi)\,\theta_m(\vartheta), \quad \Psi(\varphi, \vartheta) = \sum_{km} \Psi_{km} U_{km}(\varphi)\,\theta_m(\vartheta)$$

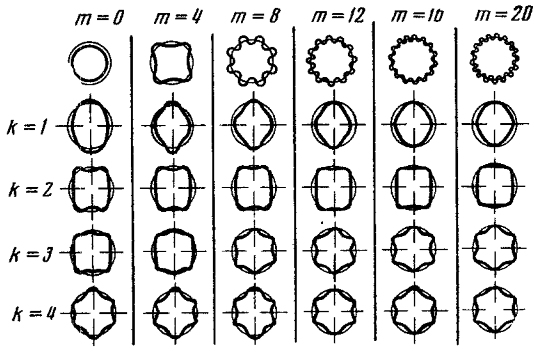


Fig. 1

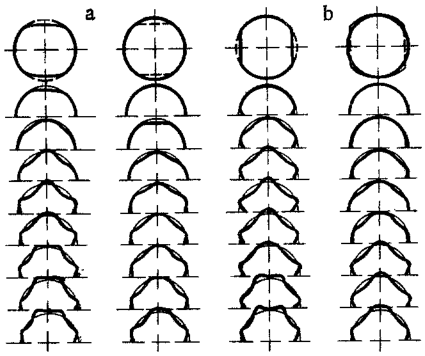


Fig. 2

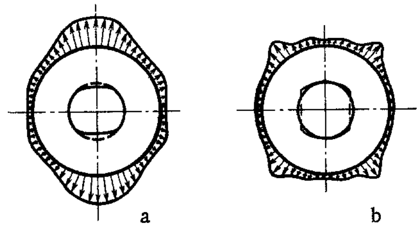


Fig. 3

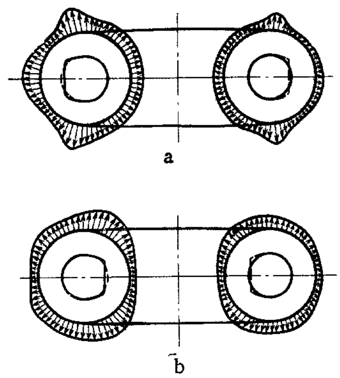


Fig. 4

$$\theta_m(\vartheta) = \begin{pmatrix} \cos m\vartheta & 0 & 0 \\ 0 & \sin m\vartheta & 0 \\ 0 & 0 & \cos m\vartheta \end{pmatrix}$$

In Fig. 2a, b we have shown the form of the meridian at successive time instants (from top to bottom) for a spherical shell, when the initial displacement or the initial velocity is localized in the neighborhood of the pole (a), or in the neighborhood of the equator (b). In the upper part of the figure we have represented the shape of the initial displacement (the solid line) or the graph of initial velocities (the dotted line). The magnitude of the initial velocity is set off along the normal from the neutral position of the meridian.

In Fig. 3a, b we have depicted membrane stresses, maximum over the time, for the spherical shell. They are also set off along the normal from the meridian. Form of the initial displacement or the graph of the initial velocities are represented inside each section (a concentric circle of a smaller radius). It should be noted that, just as in the case of localization of the initial displacement or the initial velocity close to the pole, so also in the case of localization of them in the vicinity of the equator the dynamic stresses reach extremal values at the pole.

In Fig. 3b we have depicted values of moments, maximum over the time, for the spherical shell. These stresses attain maximum values in medium latitudes ($\approx \pi/4$). The maximum zone is clearly expressed (see Fig. 3b).

Analogous stresses and moments for a toroidal shell with the parameters $R = 200$, $a = 100$, $h = 1$, $\sigma = 0.25$ are shown in Fig. 4a, b respectively. The left sides of the figures correspond to cases in which initial velocities are absent, while the right sides correspond to the cases where initial displacements are absent. We note that independently of the method of initial perturbation (using the positive, zero or negative Gaussian curvature) the extremal values of the membrane stresses and moments in the toroidal shell are reached close to the line where the sign of the Gaussian curvature changes.

The results presented confirm the possibility of an effective calculation study of the dynamic characteristic according to the procedure proposed in [1], by means of the Green matrices computed beforehand.

## LITERATURE CITED


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